

Vanishing results for the Aomoto complex of real hyperplane arrangements via minimality

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December 17, 2015

Abstract

We prove vanishing results of the cohomology groups of Aomoto complex over arbitrary coefficient ring for real hyperplane arrangements. The proof is using minimality of arrangements and descriptions of Aomoto complex in terms of chambers.

Our methods also provide a new proof for the vanishing theorem of local system cohomology groups which was first proved by Cohen, Dimca and Orlik.

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1 Introduction

Theory of hypergeometric integrals originated from Gauss has been generalized to higher dimensions, which has applications in various area of mathematics and physics ([1, 8, 15]). In the above generalization, the notion of the local system cohomology groups on the complement of a hyperplane arrangement plays a crucial role.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine hyperplanes in \mathbb{C}^ℓ , $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ be its complement. We also fix a defining equation α_i of H_i . An arrangement \mathcal{A} is called essential if normal vectors of hyperplanes generate \mathbb{C}^ℓ . The first homology group $H_1(M(\mathcal{A}), \mathbb{Z})$ is a free abelian group generated by the meridians $\gamma_1, \dots, \gamma_n$ of hyperplanes. We denote their dual basis by $e_1, \dots, e_n \in H^1(M(\mathcal{A}), \mathbb{Z})$. The element e_i can be identified with $\frac{1}{2\pi\sqrt{-1}}d \log \alpha_i$ via the de Rham isomorphism.

The isomorphism class of a rank one complex local system \mathcal{L} is determined by a homomorphism $\rho : H_1(M(\mathcal{A}), \mathbb{Z}) \longrightarrow \mathbb{C}^\times$, which is also determined by an n -tuple $q = (q_1, \dots, q_n) \in (\mathbb{C}^\times)^n$, where $q_i = \rho(\gamma_i)$.

For a generic parameter (q_1, \dots, q_n) , it is known that the following vanishing result holds.

$$\dim H^k(M(\mathcal{A}), \mathcal{L}) = \begin{cases} 0, & \text{if } k \neq \ell, \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \quad (1)$$

Several sufficient conditions for the vanishing (1) have been known ([1, 7]). Among others, Cohen, Dimca and Orlik ([3]) proved the following.

Theorem 1.1. (*CDO-type vanishing theorem*) *Suppose that $q_X \neq 1$ for each dense edge X contained in the hyperplane at infinity. Then the vanishing (1) holds. (See §2.1 below for terminologies).*

The above result is stronger than many other vanishing results. Indeed for the case $\ell = 2$, it was proved in [17] that the vanishing (1) with additional property holds if and only if the assumption of Theorem 1.1 holds.

The local system cohomology group $H^k(M(\mathcal{A}), \mathcal{L})$ is computed by using twisted de Rham complex $(\Omega_{M(\mathcal{A})}^\bullet, d + \omega \wedge)$, with $\omega = \sum \lambda_i d \log \alpha_i$, where λ is

a complex number such that $\exp(-2\pi\sqrt{-1}\lambda_i) = q_i$. The algebra of rational differential forms $\Omega_{M(\mathcal{A})}^\bullet$ has a natural \mathbb{C} -subalgebra $A_{\mathbb{C}}^\bullet(\mathcal{A})$ generated by $e_i = \frac{1}{2\pi\sqrt{-1}}d\log \alpha_i$. This subalgebra is known to be isomorphic to the cohomology ring $H^\bullet(M(\mathcal{A}), \mathbb{C})$ of $M(\mathcal{A})$ ([2]) and having a combinatorial description the so-called Orlik-Solomon algebra [10] (see §2.1 below for details). The Orlik-Solomon algebra provides a subcomplex $(A_{\mathbb{C}}^\bullet(\mathcal{A}), \omega \wedge)$ of the twisted de Rham complex, which is called the Aomoto complex. There exists a natural morphism

$$(A_{\mathbb{C}}^\bullet(\mathcal{A}), \omega \wedge) \hookrightarrow (\Omega_{M(\mathcal{A})}^\bullet, d + \omega \wedge) \quad (2)$$

of complexes. The Aomoto complex $(A_{\mathbb{C}}^\bullet(\mathcal{A}), \omega \wedge)$ has a purely combinatorial description. Furthermore, it can be considered as a linearization of the twisted de Rham complex $(\Omega_{M(\mathcal{A})}^\bullet, d + \omega \wedge)$. Indeed, there exists a Zariski open subset $U \subset (\mathbb{C}^\times)^n$ which contains $(1, 1, \dots, 1) \in (\mathbb{C}^\times)^n$ such that (2) is quasi-isomorphic for $q \in U$ ([6, 14, 9]). However, they are not isomorphic in general.

Vanishing results for the cohomology of the Aomoto complex are also proved by Yuzvinsky.

Theorem 1.2. ([19, 20]) *Let $\omega = \sum_{i=1}^n 2\pi\sqrt{-1}\lambda_i e_i \in A_{\mathbb{C}}^1(\mathcal{A})$. Suppose $\lambda_X \neq 0$ for all dense edge X in $L(\mathcal{A})$. Then we have*

$$\dim H^k(A_{\mathbb{C}}^\bullet(\mathcal{A}), \omega \wedge) = \begin{cases} 0, & \text{if } k \neq \ell, \\ |\chi(M(\mathcal{A}))|, & \text{if } k = \ell. \end{cases} \quad (3)$$

We note that the assumptions in Theorem 1.1 and Theorem 1.2 are somewhat complementary. For the first one requires nonresonant condition along the hyperplane at infinity, on the other hand, Theorem 1.2 imposes nonresonant condition on all dense edges in the affine space.

Recently, Papadima and Suciu proved that for a torsion local system, the dimension of the local system cohomology group is bounded by that of Aomoto complex with finite field coefficients.

Theorem 1.3. ([13]) *Let $p \in \mathbb{Z}$ be a prime. Suppose $\omega = \sum_{i=1}^n \lambda_i e_i \in A_{\mathbb{F}_p}^1(\mathcal{A})$ and \mathcal{L} is the local system determined by $q_i = \exp(\frac{2\pi\sqrt{-1}}{p}\lambda_i)$. Then*

$$\dim_{\mathbb{C}} H^k(M(\mathcal{A}), \mathcal{L}) \leq \dim_{\mathbb{F}_p} H^k(A_{\mathbb{F}_p}^\bullet(\mathcal{A}), \omega \wedge), \quad (4)$$

for all $k \geq 0$.

In view of Papadima and Suciu's inequality (4), it is natural to expect that CDO-type vanishing theorem for a p -torsion local system may be deduced

from that of the Aomoto complex with finite field coefficients. The main result of this paper is the following CDO-type vanishing theorem for Aomoto complex with arbitrary coefficient ring.

Theorem 1.4. *(Theorem 3.1) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential affine hyperplane arrangement in \mathbb{R}^ℓ . Let R be a commutative ring with 1. Let $\omega = \sum_{i=1}^n \lambda_i e_i \in A_R^1(\mathcal{A})$. Suppose that $\lambda_X \in R^\times$ for any dense edge X contained in the hyperplane at infinity. Then the following holds.*

$$H^k(A_R^\bullet(\mathcal{A}), \omega \wedge) \simeq \begin{cases} 0, & \text{if } k \neq \ell, \\ R^{|\chi(M(\mathcal{A}))|}, & \text{if } k = \ell. \end{cases} \quad (5)$$

Our proof relies on several works ([16, 17, 18]) concerning minimality of arrangements. We can also provide an alternative proof of Theorem 1.1 for real arrangements.

This paper is organized as follows.

In §2, we recall basic terminologies and the description of Aomoto complex in terms of chambers developed in [16, 17, 18]. We also recall the description of twisted minimal complex in terms of chambers. Simply speaking, two cochain complexes $(R[\text{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ and $(\mathbb{C}[\text{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ are constructed by using the real structures of \mathcal{A} (adjacent relations of chambers). These cochain complexes provide a parallel description between the cohomology of Aomoto complex and the local system cohomology group. Indeed, using these complexes, we can prove simultaneously CDO-type vanishing result for both cases.

In §3, we state the main result and describe the strategy for the proof. The proof consists of an easy part and a hard part. The easy part of the proof is done mainly by elementary arguments on cochain complex, which is also done in this section. The hard part is done in the subsequent section (§4).

The final section §4 is devoted to analyze the polyhedral structures of chambers which are required for matrix presentations of the coboundary map of $(R[\text{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$.

2 Notations and Preliminaries

2.1 Orlik-Solomon algebra and Aomoto complex

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine hyperplane arrangement in $V = \mathbb{R}^\ell$. Denote by $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \cup_{i=1}^n H_i \otimes \mathbb{C}$ the complement of the complexified

hyperplanes. By identifying \mathbb{R}^ℓ with $\mathbb{P}_{\mathbb{R}}^\ell \setminus \overline{H}_\infty$, define the projective closure by $\overline{\mathcal{A}} = \{\overline{H}_1, \dots, \overline{H}_n, \overline{H}_\infty\}$, where $\overline{H}_i \subset \mathbb{P}_{\mathbb{R}}^\ell$ is the closure of H_i in the projective space. We denote $L(\mathcal{A})$ and $L(\overline{\mathcal{A}})$ the intersection posets of \mathcal{A} and $\overline{\mathcal{A}}$, respectively, namely, the poset of subspaces obtained as intersections of some hyperplanes with reverse inclusion order. An element of $L(\mathcal{A})$ (and $L(\overline{\mathcal{A}})$) is also called an edge. We denote by $L_k(\mathcal{A})$ the set of all k -dimensional edges. For example $L_\ell(\mathcal{A}) = \{V\}$ and $L_{\ell-1}(\mathcal{A}) = \mathcal{A}$. Then \mathcal{A} is essential if and only if $L_0(\mathcal{A}) \neq \emptyset$.

Let R be a commutative ring. Orlik and Solomon gave a simple combinatorial description of the algebra $H^*(M(\mathcal{A}), R)$, which is the quotient of the exterior algebra on classes dual to the meridians, modulo a certain ideal determined by $L(\mathcal{A})$, see [10]. More precisely, by associating to any hyperplane H_i a generator $e_i \simeq \frac{1}{2\pi\sqrt{-1}} d \log \alpha_i$, the Orlik-Solomon algebra $A_R^\bullet(\mathcal{A})$ of \mathcal{A} is the quotient of the exterior algebra generated by the elements e_i , $1 \leq i \leq n$, modulo the ideal $I(\mathcal{A})$ generated by:

- the elements of the form $\{e_{i_1} \wedge \dots \wedge e_{i_s} \mid H_{i_1} \cap \dots \cap H_{i_s} = \emptyset\}$,
- the elements of the form $\{\partial(e_{i_1} \wedge \dots \wedge e_{i_s}) \mid H_{i_1} \cap \dots \cap H_{i_s} \neq \emptyset \text{ and } \text{codim}(H_{i_1} \cap \dots \cap H_{i_s}) < s\}$, where $\partial(e_{i_1} \wedge \dots \wedge e_{i_s}) = \sum_{\alpha=1}^s (-1)^{\alpha-1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_\alpha}} \wedge \dots \wedge e_{i_s}$.

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in R^n$ and $\omega_\lambda = \sum_{i=1}^n \lambda_i e_i \in A_R^1(\mathcal{A})$. The cochain complex $(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge) = \{A_R^\bullet(\mathcal{A}) \xrightarrow{\omega_\lambda \wedge} A_R^{\bullet+1}(\mathcal{A})\}$ is called *the Aomoto complex*.

We say that an edge $X \in L(\overline{\mathcal{A}})$ is *dense* if the localization $\overline{\mathcal{A}}_X = \{\overline{H} \in \overline{\mathcal{A}} \mid X \subseteq \overline{H}\}$ is indecomposable (see [12] for more details). We consider each hyperplane $\overline{H} \in \overline{\mathcal{A}}$ is a dense edge. In this paper, the set of dense edges of $\overline{\mathcal{A}}$ contained in \overline{H}_∞ plays an important role. We denote by $D_\infty(\overline{\mathcal{A}})$ the set of all dense edges contained in \overline{H}_∞ . We will characterize $X \in D_\infty(\overline{\mathcal{A}})$ in terms of chambers in Proposition 2.6.

Set $\lambda_\infty := -\sum_{i=1}^n \lambda_i$, and for any $X \in L(\overline{\mathcal{A}})$, $\lambda_X := \sum_{\overline{H}_i \supset X} \lambda_i$, where the index i runs $\{1, 2, \dots, n, \infty\}$.

The isomorphism class of a rank one local system \mathcal{L} on the complexified complement $M(\mathcal{A})$ is determined by the monodromy $q_i \in \mathbb{C}^\times$ around each hyperplane H_i . As in the case of Aomoto complex, we denote $q_\infty = (q_1 q_2 \dots q_n)^{-1}$ and $q_X = \prod_{\overline{H}_i \supset X} q_i$ for an edge $X \in L(\overline{\mathcal{A}})$.

2.2 Chambers and minimal complex

In this section, we recall the description of the minimal complex in terms of real structures from [16, 17, 18]. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential hyperplane arrangement in \mathbb{R}^ℓ . A connected component of $\mathbb{R}^\ell \setminus \bigcup_{i=1}^n H_i$ is

called a chamber. The set of all chambers of \mathcal{A} is denoted by $\text{ch}(\mathcal{A})$. A chamber $C \in \text{ch}(\mathcal{A})$ is called a bounded chamber if C is bounded. The set of all bounded chambers of \mathcal{A} is denoted by $\text{bch}(\mathcal{A})$. For a chamber $C \in \text{ch}(\mathcal{A})$, denote by \overline{C} the closure of C in $\mathbb{P}_{\mathbb{R}}^{\ell}$. It is easily seen that a chamber C is bounded if and only if $\overline{C} \cap \overline{H}_{\infty} = \emptyset$.

For given two chambers $C, C' \in \text{ch}(\mathcal{A})$, denote by

$$\text{Sep}(C, C') := \{H_i \in \mathcal{A} \mid H_i \text{ separates } C \text{ and } C'\},$$

the set of separating hyperplanes of C and C' .

For the description of the minimal complex, we have to fix a generic flag. Let

$$\mathcal{F} : \emptyset = F^{-1} \subset F^0 \subset F^1 \subset \dots \subset F^{\ell} = \mathbb{R}^{\ell}$$

be a generic flag (i.e., F^k is a generic k -dimensional affine subspace, in other words, $\dim(\overline{X} \cap \overline{F}^k) = \dim \overline{X} + k - \ell$ for any $\overline{X} \in L(\overline{\mathcal{A}})$). The genericity of \mathcal{F} is equivalent to

$$F^k \cap L_i(\mathcal{A}) = L_{k+i-\ell}(\mathcal{A} \cap F^k),$$

for $k + i \geq \ell$.

Definition 2.1. We say that the hyperplane $F^{\ell-1}$ is near to \overline{H}_{∞} when $F^{\ell-1}$ does not separate 0-dimensional edges $L_0(\mathcal{A}) \subset \mathbb{R}^{\ell}$. Similarly, we say the flag \mathcal{F} is near to \overline{H}_{∞} when F^{k-1} does not separate $L_0(\mathcal{A} \cap F^k)$ for all $k = 1, \dots, \ell$.

From this point, we assume that the flag \mathcal{F} is near to \overline{H}_{∞} . For a generic flag \mathcal{F} near to \overline{H}_{∞} , we define

$$\begin{aligned} \text{ch}^k(\mathcal{A}) &= \{C \in \text{ch}(\mathcal{A}) \mid C \cap F^k \neq \emptyset, C \cap F^{k-1} = \emptyset\} \\ \text{bch}^k(\mathcal{A}) &= \{C \in \text{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is bounded}\} \\ \text{uch}^k(\mathcal{A}) &= \{C \in \text{ch}^k(\mathcal{A}) \mid C \cap F^k \text{ is unbounded}\}. \end{aligned}$$

Then clearly, we have

$$\begin{aligned} \text{ch}^k(\mathcal{A}) &= \text{bch}^k(\mathcal{A}) \sqcup \text{uch}^k(\mathcal{A}) \\ \text{ch}(\mathcal{A}) &= \bigsqcup_{k=0}^{\ell} \text{ch}^k(\mathcal{A}). \end{aligned}$$

Note that $\text{bch}^{\ell}(\mathcal{A}) = \text{bch}(\mathcal{A})$, however, for $k < \ell$, $C \in \text{bch}^k(\mathcal{A})$ is an unbounded chamber.

Definition 2.2. ([17, Definition 2.1]) Let $C \in \text{bch}(\mathcal{A})$. There exists a unique chamber, denoted by $C^{\vee} \in \text{uch}(\mathcal{A})$, which is the opposite with respect to $\overline{C} \cap \overline{H}_{\infty}$, where \overline{C} is the closure of C in the projective space $\mathbb{P}_{\mathbb{R}}^{\ell}$.

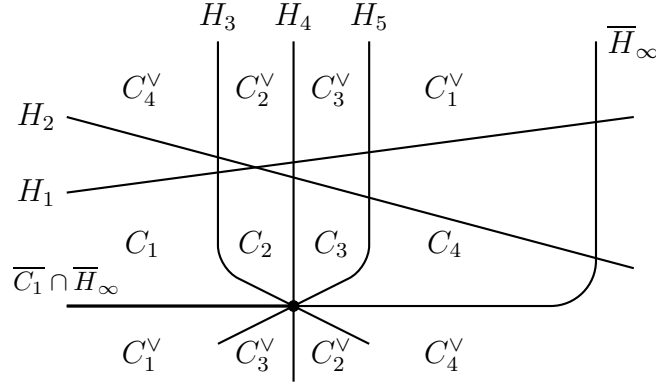


Figure 1: Opposite chambers

Let us denote the projective subspace generated by $\overline{C} \cap \overline{H}_\infty$ by $X(C) = \langle \overline{C} \cap \overline{H}_\infty \rangle$.

Proposition 2.3. *Let $C \in \text{bch}(\mathcal{A})$, then*

$$\text{Sep}(C, C^\vee) = \{H \in \mathcal{A} \mid \overline{H} \not\supset X(C)\} = \overline{\mathcal{A}} \setminus \overline{\mathcal{A}}_{X(C)}. \quad (6)$$

Proof. Let $p \in C$ and p' be a point in the relative interior of $\overline{C} \cap \overline{H}_\infty$. Take the line $L = \langle p, p' \rangle \subset \mathbb{P}_\mathbb{R}^\ell$. Choose a point $p'' \in C^\vee \cap L$. Then consider the segment $[p, p''] \subset \mathbb{R}^\ell = \mathbb{P}_\mathbb{R}^\ell \setminus \overline{H}_\infty$ (See Figure 2). On the projective space $\mathbb{P}_\mathbb{R}^\ell$, the line $L = \langle p, p' \rangle$ must intersect every hyperplane $\overline{H} \in \overline{\mathcal{A}}$ exactly once. Furthermore, L intersects $\overline{H} \in \overline{\mathcal{A}}_{X(C)}$ at p' . On the other hand, the segment $[p, p'']$ intersects $H \in \text{Sep}(C, C^\vee)$. Hence we have (6). \square

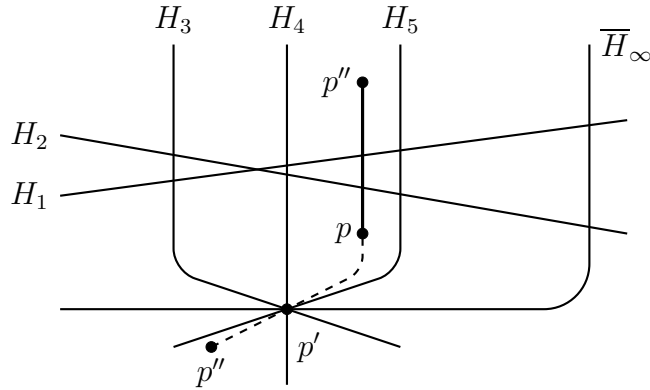


Figure 2: The segment $[p, p'']$ (thick segment).

Corollary 2.4. *If $\dim X(C) = \ell - 1$, then $\text{Sep}(C, C^\vee) = \mathcal{A}$.*

Proof. In this case, $\overline{\mathcal{A}}_{X(C)} = \{\overline{H}_\infty\}$. Proposition 2.3 concludes $\text{Sep}(C, C^\vee) = \mathcal{A}$. \square

Proposition 2.5. *([16, 17])*

- (i) $\# \text{ch}^k(\mathcal{A}) = b_k$, where $b_k = b_k(M(\mathcal{A}))$.
- (ii) $\# \text{bch}^k(\mathcal{A}) = \# \text{uch}^{k+1}(\mathcal{A})$.
- (iii) $\# \text{bch}^k(\mathcal{A}) = b_k - b_{k-1} + \cdots + (-1)^k b_0$.

Concerning (ii) of Proposition 2.5, an explicit bijection is given by the opposite chamber,

$$\iota : \text{bch}^k(\mathcal{A}) \xrightarrow{\cong} \text{uch}^{k+1}(\mathcal{A}), C \mapsto C^\vee.$$

Next result characterizes the dense edge contained in \overline{H}_∞ .

Proposition 2.6. *([17, Proposition 2.4]) Let \mathcal{A} be an affine arrangement in \mathbb{R}^ℓ . An edge $X \in L(\overline{\mathcal{A}})$ with $X \subseteq \overline{H}_\infty$ is dense if and only if $X = X(C)$ for some chamber $C \in \text{uch}(\mathcal{A})$. In particular, we have*

$$\text{D}_\infty(\mathcal{A}) = \{X(C) \mid C \in \text{uch}(\mathcal{A})\}. \quad (7)$$

Next we define the degree map

$$\text{deg} : \text{ch}^k(\mathcal{A}) \times \text{ch}^{k+1}(\mathcal{A}) \longrightarrow \mathbb{Z}.$$

Let $B = B^k \subset F^k$ be a k -dimensional ball with sufficiently large radius so that every 0-dimensional edge $X \in L_0(\mathcal{A} \cap F^k) \simeq L_{\ell-k}(\mathcal{A})$ is contained in the interior of B^k . Let $C \in \text{ch}^k(\mathcal{A})$ and $C' \in \text{ch}^{k+1}(\mathcal{A})$. Then there exists a vector field $U^{C'}$ on F^k ([16]) which satisfies the following conditions.

- $U^{C'}(x) \neq 0$ for $x \in \partial \overline{C} \cap B^k$.
- Let $x \in \partial(B^k) \cap \overline{C}$. Then $T_x(\partial B^k)$ can be considered as a hyperplane of $T_x F^k$. We impose a condition that $U^{C'}(x) \in T_x F^k$ is contained in the half space corresponding to the inside of B^k .
- If $x \in H \cap F^k$ for a hyperplane $H \in \mathcal{A}$, then $U^{C'}(x) \notin T_x(H \cap F^k)$ and is directed to the side in which C' is lying with respect to H .

When the vector field $U^{C'}$ satisfies the above conditions, we say that *the vector field $U^{C'}$ is directed to the chamber C'* . The above conditions imply that if either $x \in H \cap F^k$ or $x \in \partial B^k$, then $U^{C'}(x) \neq 0$. Thus for $C \in \text{ch}^k(\mathcal{A})$, U is not vanishing on $\partial(\overline{C} \cap B^k)$. Hence we can consider the following Gauss map.

$$\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}.$$

Fix an orientation of F^k , which induces an orientation on $\partial(\overline{C} \cap B^k)$.

Definition 2.7. Define the degree $\deg(C, C')$ between $C \in \text{ch}^k(\mathcal{A})$ and $C' \in \text{ch}^{k+1}(\mathcal{A})$ by

$$\deg(C, C') := \deg \left(\frac{U^{C'}}{|U^{C'}|} \Big|_{\partial(\overline{C} \cap B^k)} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1} \right) \in \mathbb{Z}.$$

This is independent of the choice of $U^{C'}$ ([16]).

If the vector field $U^{C'}$ does not have zeros on $\overline{C} \cap B^k$, then the Gauss map can be extended to the map $\overline{C} \cap B^k \longrightarrow S^{k-1}$. Hence $\frac{U^{C'}}{|U^{C'}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ is homotopic to a constant map. Thus we have the following.

Proposition 2.8. *If the vector field $U^{C'}$ is nowhere zero on $\overline{C} \cap B^k$, then $\deg(C, C') = 0$.*

Example 2.9. Let $p_0 \in F^k$ such that $p_0 \notin \bigcup_{H \in \mathcal{A}} H \cup \partial B^k$. Define the pointing vector field U^{p_0} by

$$U^{p_0}(x) = \overrightarrow{x; p_0} \in T_x F^k, \quad (8)$$

where $\overrightarrow{x; p_0}$ is a tangent vector at x pointing p_0 (see Figure 3). The vector field U^{p_0} is directed to the chamber which contains p_0 . Note that $U^{p_0}(x) = 0$ if and only if $x = p_0$. Hence if $p_0 \notin C \cap B^k$, the Gauss map $\frac{U^{p_0}}{|U^{p_0}|} : \partial(\overline{C} \cap B^k) \longrightarrow S^{k-1}$ has $\deg \left(\frac{U^{p_0}}{|U^{p_0}|} \right) = 0$. Otherwise, if $p_0 \in C \cap B^k$, $\deg \left(\frac{U^{p_0}}{|U^{p_0}|} \right) = (-1)^k$.

Consider the Orlik-Solomon algebra $A_R^\bullet(\mathcal{A})$ over the commutative ring R . Let $\omega_\lambda = \sum_{i=1}^n \lambda_i e_i \in A_R^1(\mathcal{A})$, ($\lambda_i \in R$). We will describe the Aomoto complex $(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge)$ in terms of chambers. For two chambers $C, C' \in \text{ch}(\mathcal{A})$, define $\lambda_{\text{Sep}(C, C')}$ by

$$\lambda_{\text{Sep}(C, C')} := \sum_{H_i \in \text{Sep}(C, C')} \lambda_i.$$

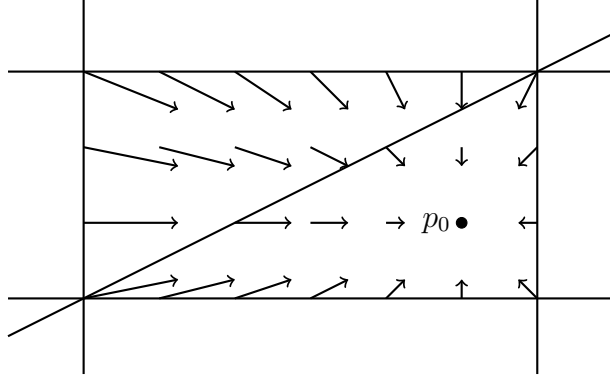


Figure 3: Pointing Vector field $\frac{1}{4}U^{p_0}$

Proposition 2.10. *Let C be an unbounded chambers. Then*

$$\lambda_{\text{Sep}(C, C^\vee)} = -\lambda_{X(C)}.$$

Proof. By Proposition 2.3, we have $\overline{\mathcal{A}} = \overline{\mathcal{A}}_{X(C)} \sqcup \text{Sep}(C, C^\vee)$. Hence, from the definition of $\lambda_\infty = -\sum_{i=1}^n \lambda_i$, we obtain $\lambda_{\text{Sep}(C, C^\vee)} + \lambda_{X(C)} = 0$. \square

Let $R[\text{ch}^k(\mathcal{A})] = \bigoplus_{C \in \text{ch}^k(\mathcal{A})} R \cdot [C]$ be the free R -module generated by $\text{ch}^k(\mathcal{A})$. Let $\nabla_{\omega_\lambda} : R[\text{ch}^k(\mathcal{A})] \rightarrow R[\text{ch}^{k+1}(\mathcal{A})]$ be the R -homomorphism defined by

$$\nabla_{\omega_\lambda}([C]) = \sum_{C' \in \text{ch}^{k+1}} \deg(C, C') \cdot \lambda_{\text{Sep}(C, C')} \cdot [C']. \quad (9)$$

Proposition 2.11. ([18]) *$(R[\text{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda})$ is a cochain complex. Furthermore, there is a natural isomorphism of cochain complexes,*

$$(R[\text{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq (A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge).$$

In particular,

$$H^k(R[\text{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge).$$

Let \mathcal{L} be a rank one local system on $M(\mathcal{A})$ which has monodromy $q_i \in \mathbb{C}^\times$ ($i = 1, \dots, n$) around H_i . Fix $q_i^{1/2} = \sqrt{q_i}$ and define $q_\infty^{1/2}$ and $\Delta(C, C')$ by $q_\infty^{1/2} := \left(q_1^{1/2} \cdots q_n^{1/2}\right)^{-1}$ and

$$\Delta(C, C') := \prod_{H_i \in \text{Sep}(C, C')} q_i^{1/2} - \prod_{H_i \in \text{Sep}(C, C')} q_i^{-1/2},$$

respectively. Then the local system cohomology group can be computed in a similar way to the Aomoto complex. Indeed, let us define the linear map $\nabla_{\mathcal{L}} : \mathbb{C}[\mathrm{ch}^k(\mathcal{A})] \longrightarrow \mathbb{C}[\mathrm{ch}^{k+1}(\mathcal{A})]$ by

$$\nabla_{\mathcal{L}}([C]) = \sum_{C' \in \mathrm{ch}^{k+1}} \deg(C, C') \cdot \Delta(C, C') \cdot [C'].$$

Then we have the following.

Proposition 2.12. ([16]) $(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}})$ is a cochain complex. Furthermore, there is a natural isomorphism of cohomology groups:

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

3 Main results and strategy

3.1 Main theorems

In this section, let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in \mathbb{R}^ℓ and R be a commutative ring with 1.

Theorem 3.1. If $\lambda_X \in R^\times$ for all $X \in D_\infty(\overline{\mathcal{A}})$, then

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) \simeq \begin{cases} 0, & \text{if } k < \ell, \\ R[\mathrm{bch}(\mathcal{A})], & \text{if } k = \ell. \end{cases}$$

More generally, we can prove the following.

Corollary 3.2. Let $0 \leq p < \ell$. If $\lambda_X \in R^\times$ for all $X \in D_\infty(\overline{\mathcal{A}})$ with $\dim(X) \geq p$, then

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\omega_\lambda}) = 0, \text{ for all } 0 \leq k < \ell - p.$$

Proof. Here we give a proof of Corollary 3.2 based on the main Theorem 3.1. If we consider $\mathcal{A} \cap F^{\ell-p}$. The Orlik-Solomon algebra $A_R^\bullet(\mathcal{A} \cap F^{\ell-p})$ is isomorphic to $A_R^{\leq \ell-p}(\mathcal{A})$. Hence we have an isomorphism

$$H^k(A_R^\bullet(\mathcal{A} \cap F^{\ell-p}), \omega_\lambda \wedge) \simeq H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge), \quad (10)$$

for $k < \ell - p$. Note that $L(\mathcal{A} \cap F^{\ell-p}) \simeq L^{\geq p}(\mathcal{A})$. By the assumption, we have $\lambda_X \in R^\times$ for any $X \in D_\infty(\mathcal{A} \cap F^{\ell-p})$. Hence by Theorem 3.1, the left hand side of (10) is vanishing. \square

By Proposition 2.11, we have the following vanishing theorem for the Aomoto complex.

Corollary 3.3. *Let $0 \leq p < \ell$. If $\lambda_X \in R^\times$ for all $X \in D_\infty(\overline{\mathcal{A}})$ with $\dim(X) \geq p$, then*

$$H^k(A_R^\bullet(\mathcal{A}), \omega_\lambda \wedge) = 0, \text{ for all } 0 \leq k < \ell - p.$$

Remark 3.4. Completely similar proof works also for the case of local systems. Namely, if the local system \mathcal{L} satisfies that $q_X \neq 1$ for all $X \in D_\infty(\overline{\mathcal{A}})$ with $\dim(X) \geq p$, then

$$H^k(\mathbb{C}[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_{\mathcal{L}}) = 0, \text{ for all } k < \ell - p.$$

Using Proposition 2.12, this implies

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0, \text{ for all } k < \ell - p,$$

which gives an alternative proof for Theorem 1.1 by Cohen, Dimca and Orlik.

3.2 Strategy for the proof of Theorem 3.1

In order to analyze the cohomology group,

$$H^k(R[\mathrm{ch}^\bullet(\mathcal{A})], \nabla_\omega) = \frac{\ker(\nabla_\omega : R[\mathrm{ch}^k(\mathcal{A})] \longrightarrow R[\mathrm{ch}^{k+1}(\mathcal{A})])}{\mathrm{im}(\nabla_\omega : R[\mathrm{ch}^{k-1}(\mathcal{A})] \longrightarrow R[\mathrm{ch}^k(\mathcal{A})])},$$

we will use the direct decomposition $R[\mathrm{ch}^k(\mathcal{A})] = R[\mathrm{bch}^k(\mathcal{A})] \oplus R[\mathrm{uch}^k(\mathcal{A})]$, and then consider the map

$$\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \hookrightarrow R[\mathrm{ch}^k(\mathcal{A})] \xrightarrow{\nabla_\omega} R[\mathrm{ch}^{k+1}(\mathcal{A})] \twoheadrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]. \quad (11)$$

We will study the map $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$ in detail below. Recall that there is a natural bijection $\iota : \mathrm{bch}^k(\mathcal{A}) \xrightarrow{\sim} \mathrm{uch}^{k+1}(\mathcal{A})$ (see Proposition 2.5 and subsequent remarks), once we fix an ordering C_1, \dots, C_b of $\mathrm{bch}^k(\mathcal{A})$, we obtain a matrix expression of the map $\overline{\nabla}_{\omega_\lambda}$. We will prove the following.

- (i) Let $C \in \mathrm{bch}^k(\mathcal{A})$. Then $\deg(C, C^\vee) = (-1)^{\ell-1-\dim X(C)}$.
- (ii) For an appropriate ordering of $\mathrm{bch}^k(\mathcal{A}) = \{C_1, \dots, C_b\}$, the matrix expression of $\overline{\nabla}_{\omega_\lambda} : R[\mathrm{bch}^k(\mathcal{A})] \longrightarrow R[\mathrm{uch}^{k+1}(\mathcal{A})]$ becomes an upper-triangular matrix.

(iii) $\det \overline{\nabla}_\omega \in R^\times$

(iv) These imply Theorem 3.1.

(i) and (ii) will be proved in §4.

Here we prove (iii) and (iv) based on (i) and (ii). First note that from Proposition 2.10, the definition (9) of the coboundary map of the complex $(R[\text{ch}^\bullet(\mathcal{A})], \nabla_\omega)$, and uppertriangularity (ii) above, we have

$$\det \overline{\nabla}_\omega = \pm \prod_{C \in \text{bch}^k(\mathcal{A})} \deg(C, C^\vee) \lambda_{X(C)}.$$

From the assumption that $\lambda_X \in R^\times$ for $X \in D_\infty(\mathcal{A})$ (see also Proposition 2.6), we obtain (iii). Since $\overline{\nabla}_\omega : R[\text{bch}^k(\mathcal{A})] \xrightarrow{\cong} R[\text{uch}^{k+1}(\mathcal{A})]$ is an isomorphism of free R -modules, which are diagonals of the following diagram, we have $H^k(R[\text{ch}^\bullet(\mathcal{A})], \nabla_\omega) = 0$ for $k < \ell$ and $H^\ell(R[\text{ch}^\bullet(\mathcal{A})], \nabla_\omega) \simeq R[\text{bch}^\ell(\mathcal{A})]$.

$$\begin{array}{ccccccc} R[\text{ch}^0] & \xrightarrow{\nabla_\omega} & R[\text{ch}^1] & \xrightarrow{\nabla_\omega} & \cdots & \xrightarrow{\nabla_\omega} & R[\text{ch}^k] & \xrightarrow{\nabla_\omega} & R[\text{ch}^{k+1}] & \xrightarrow{\nabla_\omega} & \cdots \\ \parallel & & \parallel & & & & \parallel & & \parallel & & \\ R[\text{bch}^0] & & R[\text{bch}^1] & & \cdots & & R[\text{bch}^k] & & R[\text{bch}^{k+1}] & & \\ & \searrow & \oplus & \searrow & & \searrow & \oplus & \searrow & \oplus & \searrow & \\ & & R[\text{uch}^1] & & \cdots & & R[\text{uch}^k] & & R[\text{uch}^{k+1}] & & \end{array}$$

4 Proofs

In this section, we prove (i) and (ii) in §3.2 for $k = \ell - 1$. Namely:

- (i') For a chamber $C \in \text{bch}^{\ell-1}(\mathcal{A})$, $\deg(C, C^\vee) = (-1)^{\ell-1-\dim X(C)}$.
- (ii') For an appropriate ordering of $\{C_1, \dots, C_b\} = \text{bch}^{\ell-1}(\mathcal{A})$, the matrix expression of $\overline{\nabla}_{\omega_\lambda} : R[\text{bch}^{\ell-1}(\mathcal{A})] \longrightarrow R[\text{uch}^\ell(\mathcal{A})]$ becomes an upper-triangular matrix.

For other $k < \ell$, the assertions are proved by a similar way using the generic section by F^{k+1} (see the argument of the proof of Corollary 3.2).

4.1 Structure of Walls

For simplicity we will set $F = F^{\ell-1}$. Recall that $\text{bch}^{\ell-1}(\mathcal{A}) = \{C \in \text{ch}(\mathcal{A}) \mid C \cap F \text{ is a bounded chamber of } F \cap \mathcal{A}\}$. Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$. A hyperplane $H \in \mathcal{A}$ is said to be a wall of C if $H \cap F$ is a supporting hyperplane of a facet of $\overline{C} \cap F$. For any $C \in \text{bch}^{\ell-1}(\mathcal{A})$, we denote by $\text{Wall}(C)$ the set of all walls of C .

We divide the set of walls into two types.

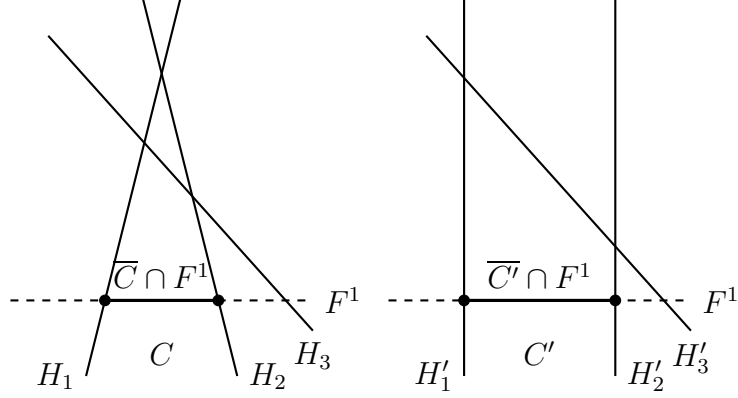


Figure 4: $\text{Wall}(C) = \text{Wall}_2(C) = \{H_1, H_2\}$, $\text{Wall}(C') = \text{Wall}_1(C') = \{H'_1, H'_2\}$

Definition 4.1. A wall $H \in \text{Wall}(C)$ is called the first kind if $\overline{H} \supset X(C)$. Otherwise H is called a wall of second kind. The set of walls of first kind, and second kind are denoted by $\text{Wall}_1(C)$ and $\text{Wall}_2(C)$ respectively. We have $\text{Wall}(C) = \text{Wall}_1(C) \sqcup \text{Wall}_2(C)$. (See Figure 4 and 5.)

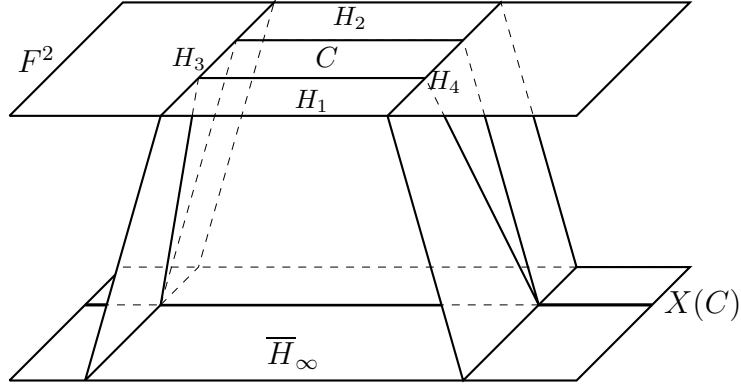


Figure 5: $\text{Wall}_1(C) = \{H_1, H_2\}$, $\text{Wall}_2(C) = \{H_3, H_4\}$.

Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$ and $\text{Wall}_1(C) = \{H_{i_1}, \dots, H_{i_k}\}$ the walls of first kind. We choose defining equations $\alpha_{i_1}, \dots, \alpha_{i_k}$ of $\text{Wall}_1(C)$ so that

$$C \subset \{\alpha_{i_1} > 0\} \cap \dots \cap \{\alpha_{i_k} > 0\}.$$

Note that $\tilde{C} := \{\alpha_{i_1} > 0\} \cap \dots \cap \{\alpha_{i_k} > 0\}$ is a chamber of $\text{Wall}_1(\mathcal{A})$. Let $D \in \text{uch}(\mathcal{A})$ be another unbounded chamber of \mathcal{A} . Then D is said to be inside $\text{Wall}_1(C)$ if

$$D \subset \tilde{C} = \{\alpha_{i_1} > 0\} \cap \dots \cap \{\alpha_{i_k} > 0\}.$$

This condition is also equivalent to $\text{Sep}(C, D) \cap \text{Wall}_1(C) = \emptyset$.

Recall that the opposite chamber of $C \in \text{bch}^{\ell-1}(\mathcal{A})$ is defined as the opposite chamber with respect to $X(C) \subset \overline{H}_\infty$. Using (6), we have the following.

Proposition 4.2. *Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$. Then $\text{Sep}(C, C^\vee) \cap \text{Wall}(C) = \text{Wall}_2(C)$.*

Remark 4.3. Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$. If D is inside the walls of $\text{Wall}_1(C)$, then we have $X(D) \subset X(C)$ and $\dim X(D) \leq \dim X(C)$.

4.2 Fibered structure of chambers

Let $d = \dim X(C)$. Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$. As above, we let $\tilde{C} \in \text{ch}(\text{Wall}_1(C))$ the unique chamber such that $C \subset \tilde{C}$.

For each point $p \in \tilde{C}$, denote by $G_1(p) := \langle X(C), p \rangle \cap F$ (Figure 6). Then $G_1(p)$ is a d -dimensional affine subspace which is parallel to each $H \in \text{Wall}_1(C)$. Fix a base point $p_0 \in \tilde{C}$. We also fix an $(\ell - 1 - d)$ -dimensional subspace $G_2(p_0) \subset F$ which is passing through p_0 and transversal to $G_1(p_0)$ (see Figure 6). Let us call $Q_0 := G_2(p_0) \cap \tilde{C}$ the base polytope.

Consider the map $\pi_C : \tilde{C} \cap F \longrightarrow Q_0, p \longmapsto G_1(p) \cap Q_0$. For each $q \in Q_0$, the fiber $\pi_C^{-1}(q) = G_1(q) \cap \tilde{C}$ is a d -dimensional polytope. This fact is a conclusion of the assumption that F is generic and near to \overline{H}_∞ , and the following elementary proposition.

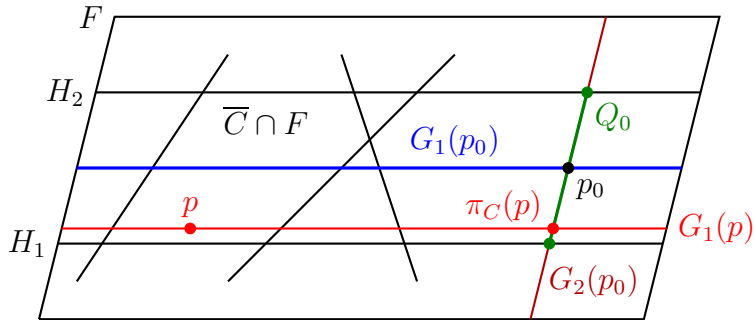


Figure 6: Base polytope Q_0 ($\text{Wall}_1(C) = \{H_1, H_2\}$)

Proposition 4.4. *Let $P \subset \mathbb{R}^\ell$ be an ℓ -dimensional polytope. Let $X \subset P$ be a d -dimensional face ($0 \leq d \leq \ell$). We denote by $\langle X \rangle$ the d -dimensional affine subspace spanned by X . Then for $\varepsilon \in \mathbb{R}^\ell$ with sufficiently small $0 \leq |\varepsilon| \ll 1$, $(\langle X \rangle + \varepsilon) \cap P$ is either an empty set or a d -dimensional polytope.*

Remark 4.5. Since $\pi_C : \overline{C} \cap F \rightarrow Q_0$ is a fibration with contractible fibers, there exists a continuous section $\sigma_C : Q_0 \rightarrow \overline{C} \cap F$ such that $\pi_C \circ \sigma_C = \text{id}_{Q_0}$.

4.3 Upper-triangularity

Let us fix an ordering of chambers of $\text{bch}^{\ell-1}(\mathcal{A}) = \{C_1, \dots, C_b\}$ in such a way that

$$\dim X(C_1) \geq \dim X(C_2) \geq \dots \geq \dim X(C_b).$$

The main result in this section is the following.

Theorem 4.6. *The matrix $(\deg(C_i, C_j^\vee))_{i,j=1,\dots,b}$ is upper-triangular. In other words, if $i > j$, $\deg(C_i, C_j^\vee) = 0$.*

Proof. Let $C, D \in \text{bch}^{\ell-1}(\mathcal{A})$. Suppose $\dim X(D) \geq \dim X(C)$ and $C \neq D$. Then we will prove $\deg(C, D^\vee) = 0$. The idea of the proof is to construct a vector field U^{D^\vee} directed to D^\vee on F which is nowhere vanishing on a neighbourhood of $\overline{C} \cap F \subset F$. Then by Proposition 2.8, we have $\deg(C, D^\vee) = 0$.

We divide into three cases.

- (a) $\dim X(C) = \ell - 1$.
- (b) $\dim X(C) < \ell - 1$ and D is not inside of $\text{Wall}_1(C)$.
- (c) $\dim X(C) < \ell - 1$ and D is inside of $\text{Wall}_1(C)$.

Firstly we consider the case (a). In this case, since $\dim X(D) \geq \dim X(C)$, we have $\dim X(D) = \ell - 1$. Choose a point $p \in D \cap F$, and define the vector field U on F by

$$U(x) = \overrightarrow{x; p} \in T_x F.$$

Then the vector field is directed to p and nowhere vanishing on $\overline{C} \cap F$ (because $p \notin \overline{C}$). By Corollary 2.4, $-U$ is a vector field directed to D^\vee , which is also nowhere vanishing on $\overline{C} \cap F$. Hence $\deg(C, D^\vee) = 0$.

From now on, we assume $\dim X(C) < \ell - 1$. If D is inside of $\text{Wall}_1(C)$, then $X(D) \subset X(C)$ by Remark 4.3, we have $\overline{\mathcal{A}}_{X(D)} \supset \overline{\mathcal{A}}_{X(C)}$. Proposition 4.2 indicates $\text{Sep}(D, D^\vee) \cap \overline{\mathcal{A}}_{X(C)} = \emptyset$. We can conclude that D^\vee is also inside $\text{Wall}_1(C)$. Conversely, if D is not inside of $\text{Wall}_1(C)$, then also D^\vee is not inside $\text{Wall}_1(C)$.

Next we consider the case (b). Then $\text{Sep}(C, D^\vee) \cap \text{Wall}_1(C) \neq \emptyset$. Choose a hyperplane $H_{i_0} \in \text{Sep}(C, D^\vee) \cap \text{Wall}_1(C)$. Let α_{i_0} be the defining equation of H_{i_0} . Without loss of generality, we may assume that

$$\begin{aligned} H_{i_0}^+ &= \{\alpha_{i_0} > 0\} \supset D^\vee \\ H_{i_0}^- &= \{\alpha_{i_0} < 0\} \supset C. \end{aligned}$$

We will construct a vector field U^{D^\vee} on F which is directed to D^\vee and satisfying

$$U^{D^\vee}(x)\alpha_{i_0} > 0, \quad (12)$$

for $x \in \overline{C} \cap F$, where the left hand side of (12) is the derivative of α_{i_0} with respect to the vector field. In particular, we obtain a vector field directed to D^\vee which is nowhere vanishing on $\overline{C} \cap F$. It is enough to show that, at any point $x_0 \in \overline{C}$, there exists a local vector field around x_0 which satisfies (12). Then we will obtain a global vector field which satisfies (12) using partition of unity.

It is sufficient to show the existence of such vector field around each vertex x_0 of $\overline{C} \cap F$. By genericity of F , $Z := \bigcap \mathcal{A}_{x_0} = \bigcap_{x_0 \in H \in \mathcal{A}} H$ is a 1-dimensional flat of \mathcal{A} , which is transversal to F . By the assumption that F does not separate 0-dimensional flats of \mathcal{A} , we have

$$\overline{Z} \cap \overline{H}_\infty \subset \overline{C} \cap \overline{H}_\infty. \quad (13)$$

(See Figure 7.)

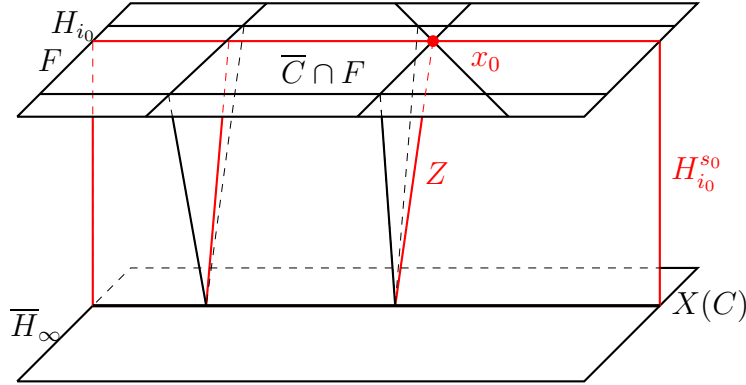


Figure 7: Z and $H_{i_0}^{s_0}$.

Set $s_0 := \alpha_{i_0}(x_0)$ and $H_{i_0}^{s_0} = \{\alpha_{i_0} = s_0\}$ the hyperplane passing through x_0 which is parallel to H_{i_0} . Then we have $Z \subset H_{i_0}^{s_0}$, otherwise, contradicting (13). The hyperplanes $\mathcal{A}_{x_0} = \mathcal{A}_Z$ determines chambers (cones), one of which, denoted by Γ , contains D^\vee (Figure 8). Hence the tangent vector $U^{D^\vee}(x_0)$ should be contained in Γ . Furthermore,

$$D \subset \Gamma \cap H_{i_0}^+ \subset \Gamma \cap H_{i_0}^{>s_0}. \quad (14)$$

In particular, we have $\Gamma \cap H_{i_0}^{>s_0} \neq \emptyset$. Thus we can construct a vector field U^{D^\vee} around x_0 so that $U^{D^\vee}(x_0) \in \Gamma \cap H_{i_0}^{>s_0}$. Then (12) is satisfied around x_0 . Hence we have $\deg(C, D^\vee) = 0$ for the case (b).

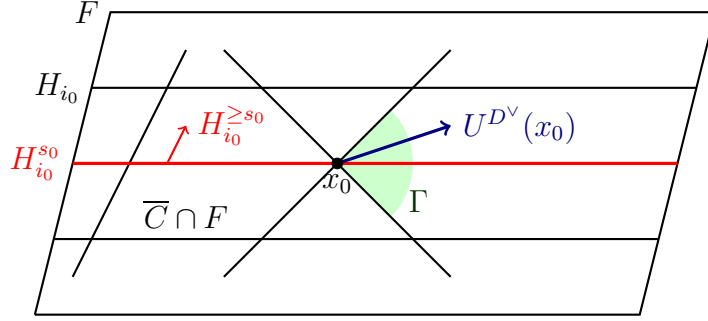


Figure 8: Construction of the vector field U^{D^\vee}

Thirdly, suppose D is inside of $\text{Wall}_1(C)$, equivalently, $D \subset \tilde{C}$. Let us handle the case (c). Since $X(D) \subset X(C)$ and $\dim X(D) \geq \dim X(C)$, we have $X(D) = X(C)$. In this case, $\text{Wall}_1(C) = \text{Wall}_1(D)$ and $\tilde{C} = \tilde{D}$. We consider the fibration $\pi_D : \overline{D} \cap F \mapsto Q_0$ which also has d -dimensional polytopes as fibers. Since the fiber is contractible, there exists a continuous section $\sigma_D : Q_0 \mapsto \overline{D} \cap F$ such that $\pi_D \circ \sigma_D = \text{id}_{Q_0}$.

Now we construct a vector field. For each $p \in \overline{C} \cap F$, we denote $G_2(p)$ the $(\ell - 1 - d)$ -dimensional subspace which is passing through p and parallel to $G_2(p_0)$ (Figure 9). Let $\{p'\} = G_2(p) \cap G_1(p_0)$. The tangent space is decomposed as $T_p F = T_p G_1(p) \oplus T_p G_2(p)$. We first construct a vector field on the second component. Let us define the tangent vector $V_2(p) \in T_p G_2(p) \subset T_p F$ by

$$V_2(p) = \overrightarrow{p; p'}. \quad (15)$$

The vector field V_2 is obviously inward with respect to $\text{Wall}_1(C)$, and vanishing on the reference fiber $G_1(p_0) \cap \overline{C}$.

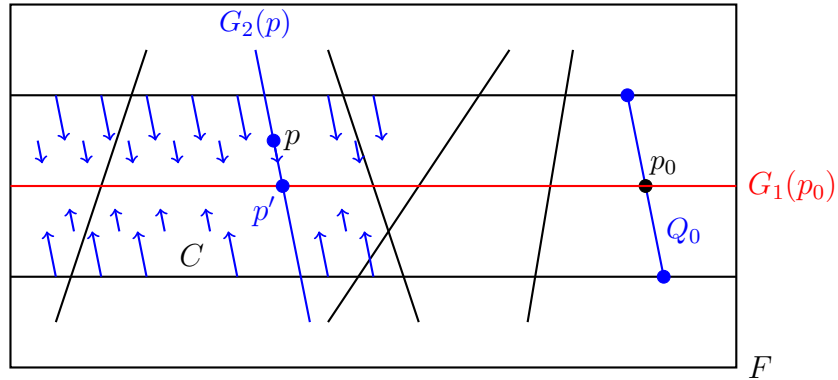


Figure 9: V_2 .

Next we construct a vector field V_1 along the fibers $G_1(p)$. Using the section $\sigma_C : Q_0 \rightarrow \overline{C} \cap F$ (Remark 4.5), define V_1 by

$$V_1(p) = \overrightarrow{p; \sigma_D(\pi_C(p))}, \quad (16)$$

(Figure 10).

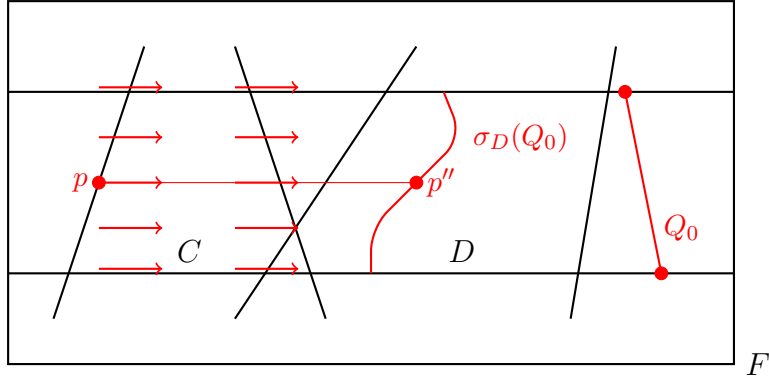


Figure 10: $V_1, p'' = \sigma_D(\pi_C(p))$.

Proposition 4.7. *For sufficiently large $t \gg 0$, the vector field $tV_1 + V_2$ is directed to D . Similarly, $-tV_1 + V_2$ is a vector field directed to D^\vee .*

Proof. Let $p \in H \in \text{Wall}_1(C)$. Recall that D is inside $\text{Wall}_1(C)$. Since V_2 is inward and V_1 is tangent to H , the vector field $\pm tV_1 + V_2$ is also inward. Let $H \in \text{Wall}_2(C)$ and $p \in H \cap F$. Then V_1 (resp. $-V_1$) is directed to D (resp. D^\vee) with respect to H . Hence for sufficiently large t , $tV_1 + V_2$ (resp. $-tV_1 + V_2$) is directed to D (resp. D^\vee). \square

Since V_1 is nowhere vanishing vector field on $\overline{C} \cap F$, $-tV_1 + V_2$ is a nowhere vanishing vector field around $\overline{C} \cap F$ which is directed to D^\vee . Hence $\deg(C, D^\vee) = 0$. This completes the proof of Theorem 4.6. \square

4.4 The degree formula

This section is devoted to prove the following.

Theorem 4.8. *Let $C \in \text{bch}^{\ell-1}(\mathcal{A})$. Suppose $\dim X(C) = d$. Then*

$$\deg(C, C^\vee) = (-1)^{\ell-1-d}. \quad (17)$$

We construct a vector field around $\overline{C} \cap F$ which is directed to C^\vee . The vector field V_2 is the same as in the previous section (§4.3). Define the vector field V_1 along the fibers π_C by

$$V_1(p) = \overrightarrow{p; \sigma_C(\pi_C(p))} \quad (18)$$

(see Figure 11).

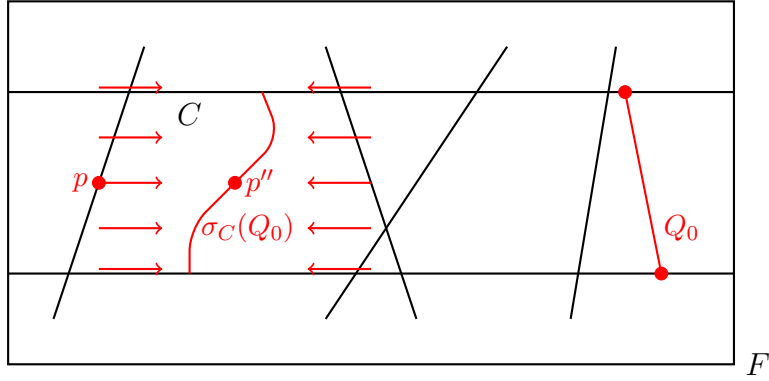


Figure 11: $V_1, p'' = \sigma_C(\pi_C(p))$.

Then $tV_1 + V_2$ is a vector field directed to C (for $t \gg 0$). Since C and C^\vee are separated by $H \in \mathcal{A} \setminus \text{Wall}_1(C)$, the vector field $-tV_1 + V_2$ is directed to C^\vee . We can compute degree $\deg(C, C^\vee)$ using the vector field $-tV_1 + V_2$. Note that $-tV_1(p)$ is outward vector field in along a d -dimensional space $G_1(p)$ and $V_2(p)$ is inward which is tangent to a $(\ell - 1 - d)$ -dimensional space $G_2(p)$. Hence $\deg(C, C^\vee)$ is equal to the index of the following vector field in $\mathbb{R}^{\ell-1}$ at the origin.

$$V = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} - \sum_{i=d+1}^{\ell-1} x_i \frac{\partial}{\partial x_i}, \quad (19)$$

where $d = \dim X(C)$. Recall that the de Rham cohomology group $H^{\ell-1}(S^{\ell-2})$ is generated by the differential form ([4])

$$\sum_{i=1}^{\ell-1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{\ell-1}.$$

It is easily seen that the self map of $H^{\ell-1}(S^{\ell-2})$ induced by the Gauss map of the vector field (19) is equal to the multiplication by $(-1)^{\ell-1-d}$. This completes the proof of Theorem 4.8.

Acknowledgements. The present work was conducted during P. Bailet’s stay at Hokkaido University as a postdoc. She thanks Postdoctoral Fellowship for Foreign Researchers (JSPS) for financial and other supports. M. Yoshinaga is partially supported by Grant-in-Aid for Scientific Research (C) 25400060 (JSPS).

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